Income growth, total return, and their double exponential interaction

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> *The prime purpose of a business corporation is to pay dividends to its owners. A successful company is one that can pay dividends regularly and presumably increase the rate as time goes on.*
>
> (Graham B. 2008)

It is quite well-known that portfolios of stocks that persistently grow their dividend (income) outperform the benchmark over a long investment horizon. In this work, the direct contribution of the dividend growth (income growth) in the total return is proved mathematically and illustrated numerically. It is proven that the impact of dividend growth is double exponential in nature, and this effect will manifest itself as the investment horizon expands. Therefore, controlling for all other variables, investing in the highest dividend growth companies results in the largest wealth appreciation.

Introduction

Dividends and their reinvestment can considerably contribute to wealth, and yet they are mostly undervalued and under appreciated by investors. Dividend yield has historically been perceived as of secondary importance to price return in the context of a security's total return. However, dividends are often used as an approach to value a company or to screen an investment universe.

The Gordon Growth Model (GGM), introduced over a half century ago, is a well-known model based on dividend and dividend growth to predict the current price of a security (Gordon 1962). Due to its power, GGM has been studied for several decades and exposed to improvements as presented (e.g., (Irons 2014) (Belomyttseva 2016)). In addition, a lot of research has been done to predict dividend growth (e.g., (Møller 2017) (Yu D. 2023)). These are indicators of why the dividend and its growth are important.

In addition to a valuation method, dividend yield and dividend growth have also been successfully utilized as a proxy for portfolio construction (Tianying 2022). In fact, companies that can increase their dividend have a financial strength that differentiates them from the broader market. These companies should have higher quality in terms of earnings and leverage as they implicitly promised to pay dividends for foreseeable future and increase these dividends.

* indicates equal contribution.

In this paper we have proven that dividend yield and dividend growth can substantially improve the total return. As a matter of fact, dividend growth increases price returns through a double exponential impact. Therefore, dividend growth not only helps during the screening process (to value a security or to select a high-quality company), but also by its nature it can enhance the total return.

The double exponential nature of dividend growth may not be easy to digest, since the human mind always uses simple heuristics based on linear reasoning (De Langhe 2016). However, even miniscule dividends subjected to high dividend growth and a long investment horizon could potentially be a considerable wealth appreciation contributor because of the inherent double exponential nature. This is the mathematical proof for Bristol Gate's investment thesis that a portfolio of the highest dividend growers over a long investment horizon should outperform the broader market.

An initial yield of a portfolio subject to high dividend growth can become an unrealistic yield over time. Bristol Gate's innovation to remedy this constraint is to rebalance the portfolio on a given frequency. This intelligent and easy to implement mechanism effectively utilizes the double exponential nature of the dividend growth while retaining the portfolio yields within the band of realistic values. Examples are provided to conclude how this approach efficiently resolves the issue.

Methodology

The cumulative total return $\mathrm{R_t}$ up to the time t is calculated by compounding each period total returns, r_i , up to time t:

$$1 + R_t = \prod_{i=1}^{t}(1 + r_i) \tag{1}$$

where each period total return r_i is defined as:

$$1 + r_i = \frac{p_i + d_i}{p_{i-1}} \tag{2}$$

and p_i and p_{i-1} are prices at time i and $i - 1$, respectively, and d_i is dividend payment at time $i. t = i$ is the time when the dividend is paid and hence i counts the dividend payments are invested at $t = i$

Defining the price return at time i as $r_i^p = \frac{p_i}{p_{i-1}} - 1$ and dividend yield as $\gamma_i = \frac{d_i}{p_i}$, the total return at time i is calculated as:

$$1 + r_i = \left(1 + r_i^p\right)(1 + \gamma_i) \tag{3}$$

This equation illustrates that wealth is accumulated by the price appreciation and income collection, i.e., dividend yield, at any point in time.

Note 1: If at the investment inception only one share is bought at price p_0 and there is no future dividend payment, the wealth generation factor at the end of period 1 is p_1/p_0. On the other hand, if there is a dividend payment d_1 for one share at the end of period 1, and it is reinvested at the same time at price p_1, the extra shares purchased is $\gamma_1 = d_1/p_1$ (the dividend payment and share repurchase are mathematically equivalent). Therefore, the total return at the end of period 1 is the combination of the new additional shares and its associated price appreciation. This could be interpreted as if $1 + \gamma_1$ shares are invested initially at price p_0 which is appreciated at the end of period 1 to price p_1; this intuition behind Equation (3) is also illustrated in Figure 1.

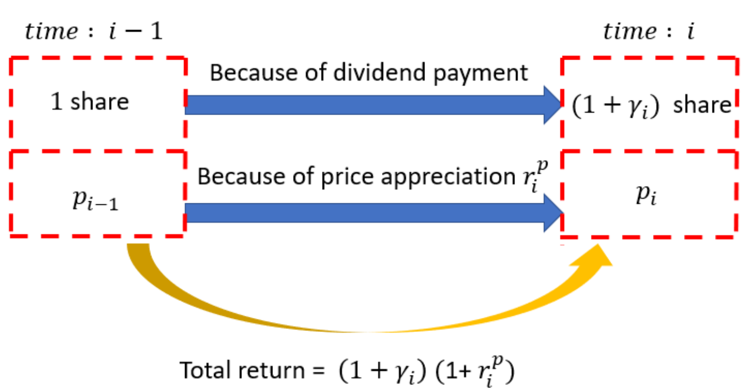

Figure 1: Additional Share because of the reinvested dividend payment and its effect on the total return

Substituting Equation (3) into Equation (1) results in:

$$1 + R_t = \left(1 + R_t^p\right)\prod_{i=1}^{t}(1 + \gamma_i) \tag{4}$$

Note 2: At the end of period 1, the total number of shares is $1 + \gamma_1$ (See Note 1). If the investment is held for two periods, at the end of period 2, the price is p_2 and the dividend amount is d_2. This dividend invested at the price p_2 generates $\gamma_2 = d_2/p_2$ extra shares. Therefore, the total effective share counts by the end of period 2 is $(1 + \gamma_1)(1 + \gamma_2)$. This process can be repeated to calculate the total effective number of shares for the entire investment horizon, which is $\prod_{i=1}^{t}(1 + \gamma_i)$

The first term on the right-hand side of Equation (4), i.e., $1 + R_t^p$, is the total compounded appreciation of \$1 solely due to price appreciation up to time t, and the second term, i.e., $\prod_{i=1}^{t}(1 + \gamma_i)$ is the total effective number of shares accumulated due to the dividend payments; refer to Note 1 and Note 2.

Since the dividend yield – or effective number of shares – at any time is much smaller than 1[1], i.e., $\gamma_i \ll 1$, using the properties of natural logarithmic function and the Taylor series expansion, the compounding of the income can be accurately approximated by:

$$\prod_{i=1}^{t}(1 + \gamma_i) \approx \exp\left(\sum_{i=1}^{t}\gamma_i\right) \tag{5}$$

Equation (5) illustrates that the inherent nature of compounding is exponential. Thus, reinvesting small gains, in this case dividend yields, can become a meaningful multiplier over time. By substituting the above equation into Equation (4), the cumulative total return is calculated as:

$$1 + R_t \approx \left(1 + R_t^p\right)\exp\left(\sum_{i=1}^{t}\gamma_i\right) \tag{6}$$

This equation demonstrates that the total cumulative wealth generation of \$1 is due to both the cumulative price appreciation and the exponential growth of the sum of the reinvested income (which could be also viewed as the total effective number of shares).

Defining income growth rate as $g_i = d_i/d_{i-1} - 1$, and the initial yield as $\gamma_0 = d_0/p_0$, the yield at time i is expressed as:

$$\gamma_i = \gamma_0 \frac{\prod_{j=1}^{i} 1 + g_j}{\prod_{j=1}^{i} 1 + r_j^p} \tag{7}$$

which illustrates that the dividend yield increases by dividend growth and shrinks by the price appreciation. Utilizing a similar approximation as in Equation (5) for both numerator and denominator in Equation (7) (for relatively small values of g_j and r_j^p), Equation (6) becomes:

$$1 + R_t \approx \left(1 + R_t^p\right)\exp\left(\gamma_0 \sum_{i=1}^{t}\exp\left(\sum_{j=1}^{i} g_j - r_j^p\right)\right) \tag{8}$$

[1] Dividend yield is generally much less than 10% and for such a range $\ln x \approx x$.

This equation[2] illustrates that the wealth appreciation is exponentially magnified by the initial yield, which on its own is exponentially scaled by dividend growth. The term $\exp\left(\gamma_0 \sum_{i=1}^{t} \exp\left(\sum_{j=1}^{i} g_j - r_j^p\right)\right)$ in Equation (8) is the effective number of shares or the contribution of the reinvested income in the total cumulative return.

The human mind can easily digest linear behaviors. However, human intuition and estimation become very inaccurate and noisy[3], when a relation manifests itself in a nonlinear fashion. The compounding effect, which was shown to be exponential, is one of those nonlinearities. The typical human's mind perceives the dividend yield as a miniscule contributor to the return and the impact of its reinvestment on wealth appreciation is generally mis-evaluated. Of more interest than the initial yield is the role of dividend growth, mainly because of its double exponential impact[4]! It is not easy to digest exponential behavior let alone double exponential behavior. However, these effects over a long investment horizon produce a remarkable difference. To have a better understanding, the difference among linear (dotted-line), exponential (dashed-line), and double exponential (solid-line) behaviours is shown in Figure 2.

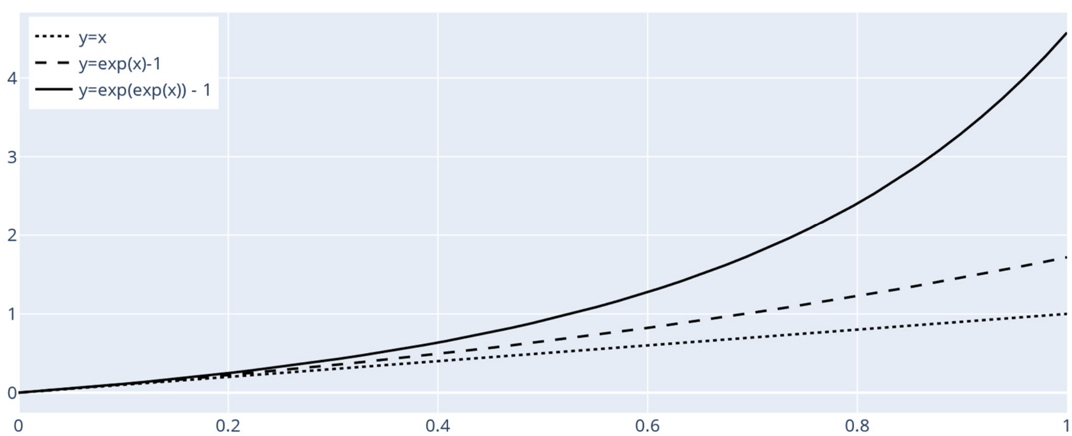

Figure 2: Difference among linear (dotted-line), exponential (dashed-line), and double exponential (solid-line) functions.

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[2] The more accurate version of this equation is $(1 + R_t^p) \exp\left(\gamma_0 \sum_{i=1}^{t} \exp\left(\sum_{j=1}^{i} c_j (g_j - r_j^p)\right)\right)$ where $c_j = 1 - g_j + r_j^p / 2$ is the correction factor due to use of higher order terms in the Taylor expansion approximation. This correction factor is not needed for the first exponential function for yield, γ, in Equation (6) since dividend yield is generally much less than 10%. However, returns and dividend growth rates can be much higher than 10% which may source inaccuracies if higher order Taylor approximations are ignored.

[3] This is the same as Weber-Fechner law in the field of psychology that relates to human perception and evaluates the connection between any improvement and the perceived reaction by people.

[4] https://en.wikipedia.org/wiki/Double_exponential_function

The terminal values in Figure 2 indicate the potential growing power of double exponential behavior compared to its counterparts from linear relation and simple exponential relation (compare 4.57 for double exponential to 1.72 and 1 for simple exponential and linear relation, respectively).

It is worth emphasizing that Equation (9) implicitly assumed that the initial dividend grows continuously with the rate g_j, at the j-th period. This assumption is violated when a portfolio is reset and/or rebalanced. This violation is because the yield of the new portfolio is not necessarily the continuation of the yield of the pre-reset/pre-rebalanced one. The approach adopted by Bristol Gate to alleviate this shortcoming is detailed in the next section, which demonstrates the innovation Bristol Gate has applied to utilize the exponential power of dividend growth while removing its bottlenecks; refer to Table 2.

Numerical Study

In this section, first it is proved that the approximation used in Equation (8) to calculate the effective number of shares, i.e., $\exp\left(\gamma_0 \sum_{i=1}^{t} \exp\left(\sum_{j=1}^{i} g_j - r_j^p\right)\right)$, is reliable and accurate. Different feasible/extreme combinations of investment horizon, dividend growth rates, dividend yield and price return are then considered, and the results are presented.

The percentage of error between the exact effective number of shares, $\prod_{i=1}^{t}(1+\gamma_i)$, and its first and second exponential approximations are presented in Table 1. The second order approximation is described in footnote 2 and considerably improves the accuracy. Errors become larger when the investment horizon is extended. Having said that, dividend growth of 10-15% over a long investment horizon is rare, if not impossible. The largest error for the second order approximation is 15.75%, which is for the case when initial yield is 4%, annual return is 4%, dividend growth is 20%, and investment horizon 15 years. It is almost impossible for a company to have an initial yield at 4% and grow its dividend at the rate of 20% for 15 years. More precisely, initial yield of 4% grows to 34% over 15 years assuming the average annualized returns and annualized dividend growth rates of 4% and 15%, respectively. This is an extreme scenario and almost impossible, and even for this extreme case the error of the approximation is only 15.75%. For feasible initial yield, annualized dividend growth rates and price returns, the results presented in Table 1 confirm that the double exponential approximation accurately represents the nature of the Equation (10).

Table 1 : % of Error between the Equation (4) – Exact – and

Equation (8) – Approximation (First order or second order)

Parameters	Approximation Order	Years of Investment		
		5	10	15
1% (initial yield) 4% (Annual return) 15% (Dividend Growth)	First order	0.26%	1.31%	4.21%
	Second order	0.03%	0.06%	0.17%
2% (yield) 4% (Annual return) 15% (Dividend Growth)	First order	0.61%	2.98%	9.70%
	Second order	0.14%	0.47%	1.34%
1% (yield) 10% (Annual return) 15% (Dividend Growth)	First order	0.13%	0.51%	1.27%
	Second order	0.02%	0.03%	0.03%
2% (yield) 10% (Annual return) 15% (Dividend Growth)	First order	0.33%	1.19%	2.89%
	Second order	0.10%	0.23%	0.38%
1% (yield) 10% (Annual return) 10% (Dividend Growth)	First order	0.02%	0.05%	0.07%
	Second order	0.02%	0.05%	0.07%
1% (yield) 10% (Annual return) 20% (Dividend Growth)	First order	0.32%	1.57%	4.83%
	Second order	0.00%	-0.06%	-0.23%
2% (yield) 10% (Annual return) 20% (Dividend Growth)	First order	0.73%	3.45%	10.71%
	Second order	0.09%	0.16%	0.29%
4% (yield) 4% (Annual return) 20% (Dividend Growth)	First order	2.82%	17.77%	90.66%
	Second order	0.75%	3.52%	15.75%

The historical medium of dividend growth and initial yield of the top quartile (top Q) as well as bottom quartile (bottom Q) of the dividend growing companies (in the S&P 500) are shown in Figure 3 and Figure 4, respectively. The median of historical dividend growth for the top Q is about 20% while this median value for the bottom Q is about 3.5%. The median of historical initial yield for the bottom Q is around 3% while this median value for the top Q is about 1.5%. Based on these historical realized values, the dividend growth rates of 15% and 5%, and initial yields of 1% and 2%, are selected in the numerical studies presented in the rest of this section.

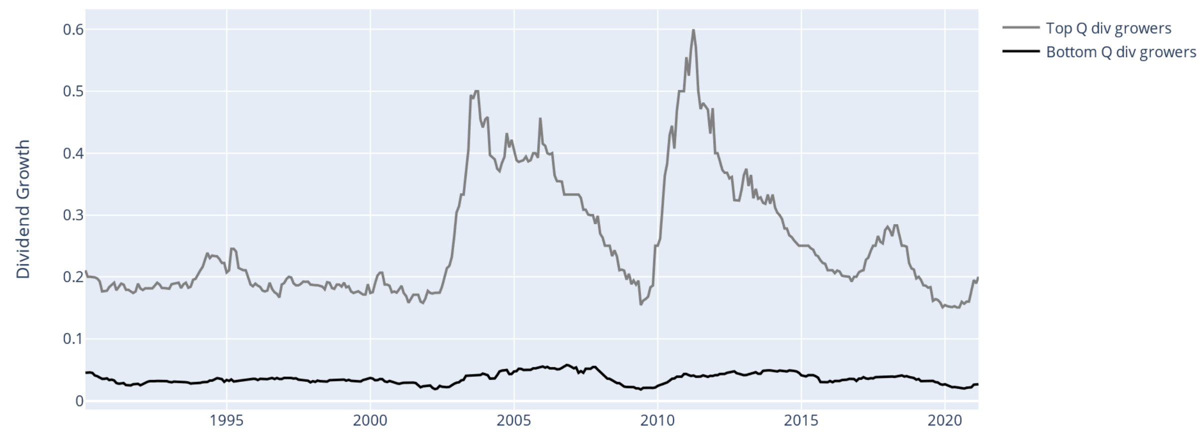


Figure 3: Historical dividend growth of the top Quartile and Bottom Quartile Dividend Growth Universe, Median

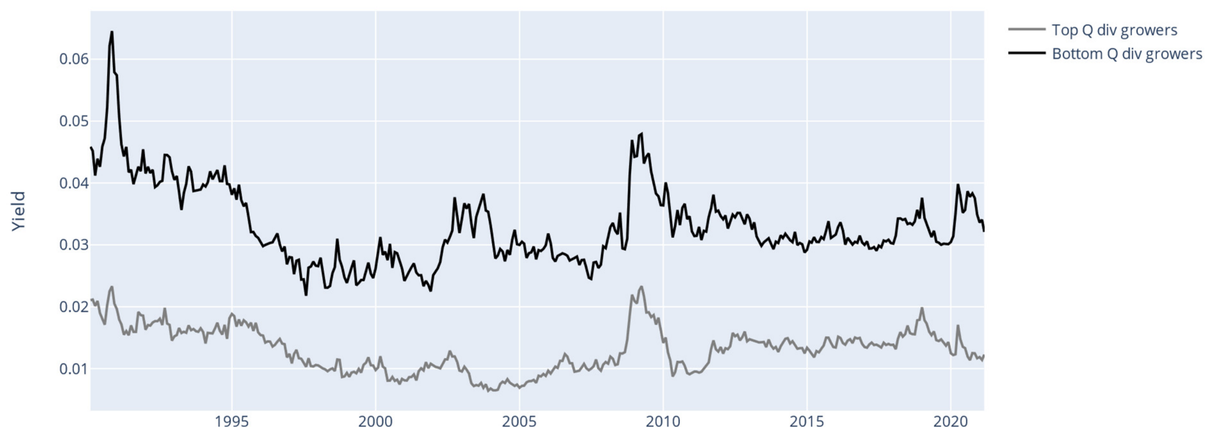


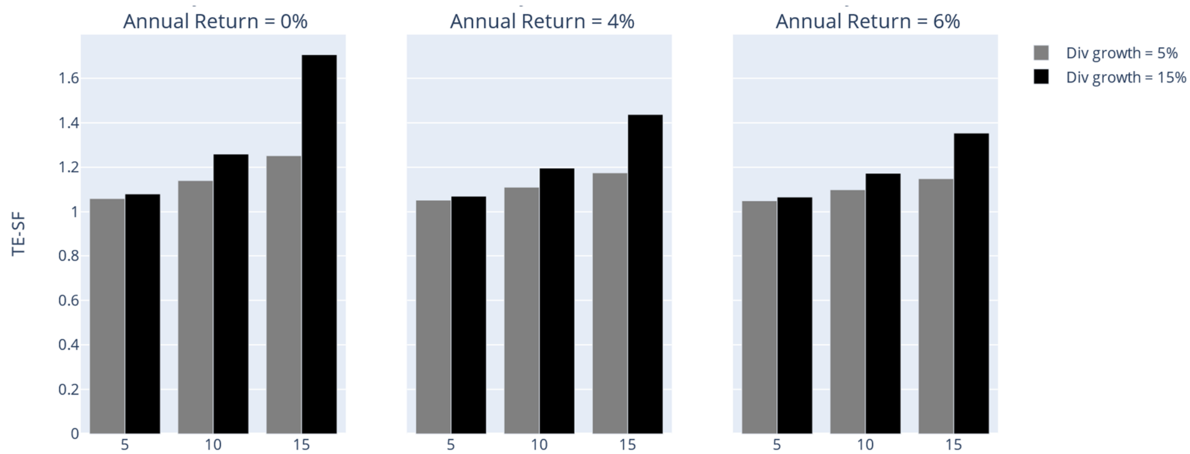
Figure 4:Historical yield of the top Quartile and Bottom Quartile Dividend Growth Universe, Median

These values are selected such that the spread between values associated with the fastest (top Q) and slowest (bottom Q) dividend growers are narrower than the realized ones. When the difference between the wealth appreciation is notable for this narrower spread (10%, which is the spread between dividend growth rates of 15% and 5%) it will be even more profound for a wider spread (i.e., 17%, which is the spread between dividend growth rates of 20% and 3%, representing historical data).

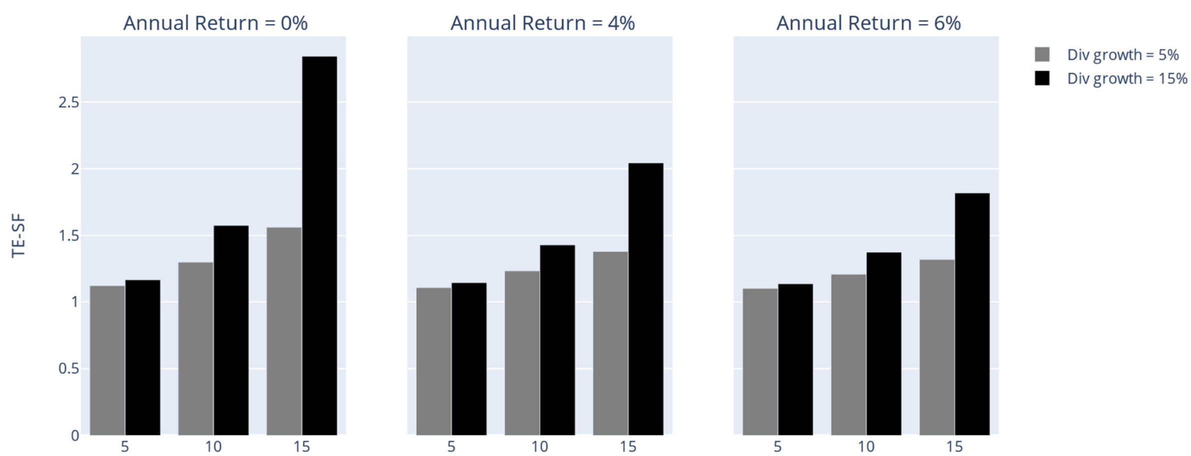
Hereafter, the total effective share factor, or dividend yield accumulation factor is referred to as "TE-SF". This is $\prod_{i=1}^{t}(1+\gamma_i)$ in Equation (4) which is approximated as exponential in Equation (11). This is the factor that the authors proved exhibits exponential characteristics and enhances the total return.

The change in TE-SF vs investment horizons for 5% dividend growth and 15% dividend growth, initial yield of 1% and 2% and moderate price returns is presented Figure 5. Faster dividend growers have considerably higher TE-SF, and longer investment horizon magnifies the TE-SF values. Of particular interest are the results when the price return is relatively small (close to zero). In this case, the fastest dividend growers outshine the slower dividend growers with the largest difference.

An initial yield of 2% and a growth rate of 15% becomes yield of 16.2%, 9.03% and 6.79% assuming price returns of 0%, 4% and 6%, respectively, after 15 years of investment. These yields are remarkably high, although not impossible. One relatively simple and yet effective solution to limit the growth of the yield is to reconstitute the portfolio when its yield reaches a given threshold.



(a) $\gamma_0 = 1\%$.



(b) $\gamma_0 = 2\%$.

Figure 5: TE-SF vs Investment time for different initial yield and dividend growths.

Longer investment horizon with higher initial yield results in higher TE-SF

Reconstituting based on yield might be complicated and hence rather than setting a hard limit on the threshold, it is possible to reconstitute the portfolio after reaching a time milestone. As an example, a portfolio of fastest dividend growers can be reconstituted every 5 years such that its yield becomes 2%.

Over a long investment horizon, a portfolio subjected to such rebalancing still has a large TE-SF while its yield remains within the feasible range; see Table 2. As illustrated in this table, assuming a price return of 6% and dividend growth of 15%, the reconstitution of the portfolio every 5 years results in slight degradation of the TE-SF from 1.82 to 1.46. However, the reconstitution/rebalancing results in a maximum achievable yield of 3% vs rarely possible yield of 6.79% (over 15 years).

Table 2: TE-SF for 15-year vs 3 re-sets each 5 years; Initial yield 2%, dividend growth = 15%.

Price Return	Year of Investment	TE-SF	Final Yield
0	15 Reset every 5 years	1.58	4.02%
4%		1.5	3.31%
6%		1.46	3.01%
0	15 Without resets	2.84	16.27%
4%		2.04	9.04%
6%		1.82	6.79%

The change in the TE-SF for different initial yields and two different regimes of dividend growth, 5% and 15% is given in Figure 6. The results in this figure are for two different investment horizons and different combinations of price returns. Higher price return results in smaller TE-SF, all else equal, as the higher price returns increase the price and apply a downward pressure on the number of purchased shares by the received dividend. A longer horizon allows the effective shares to compound longer and hence leads to higher TE-SF. Finally, higher initial yield means a higher terminal TE-SF, since the initial larger dividend payment eventually translates to higher terminal lump sum values.

(a) $n = 5\ years$

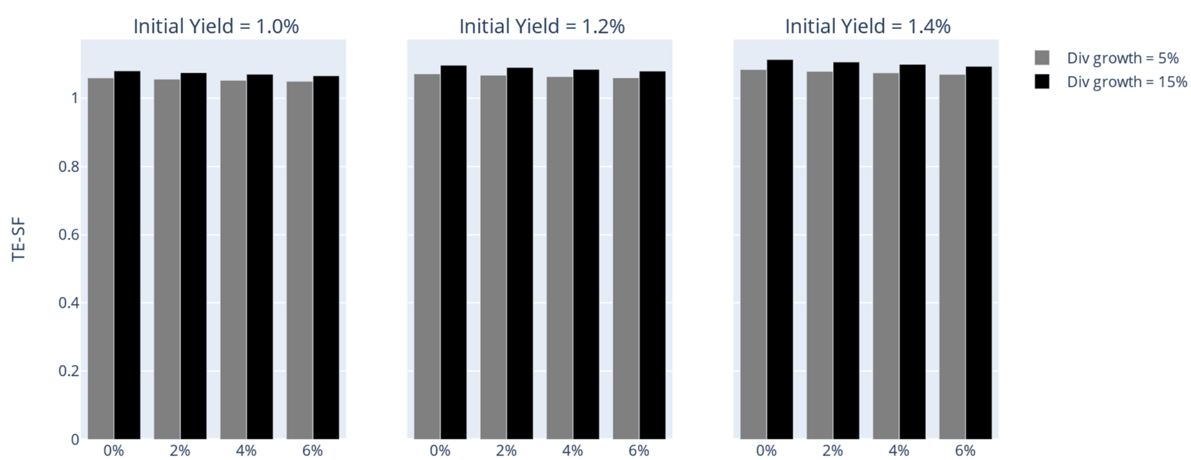
(b) $n = 10\ years$

Figure 6: TE-SF vs initial yield for different dividend growth and investment horizon.

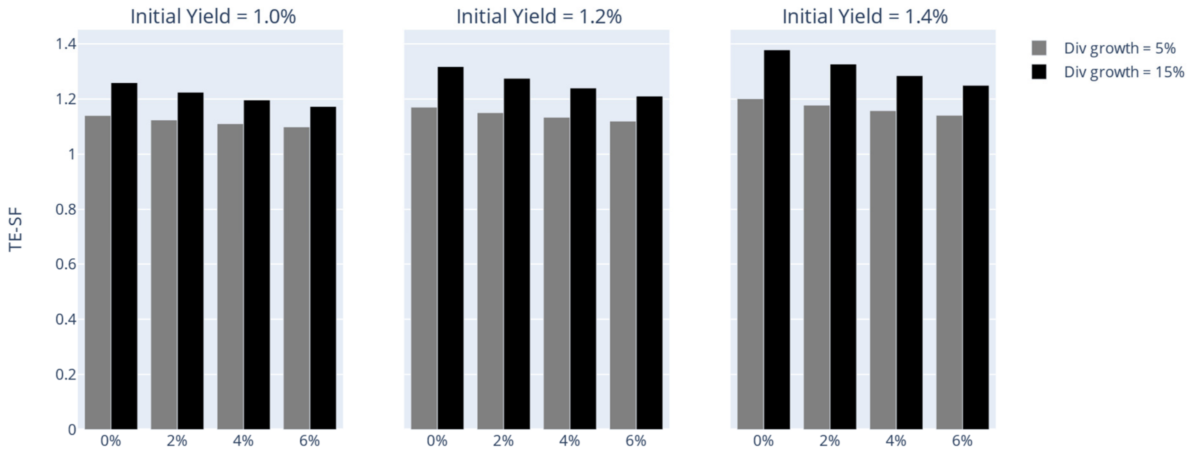
Higher initial yield increases the TF-SF and larger investment horizon cooperate with the higher initial yield

The change in TE-SF due to price return is presented in Figure 7. Higher price returns equate to smaller TE-SF all else equal because higher prices reduce the share repurchase power of the dividends.

Based on all the graphs presented here, initial yield and investment horizon are positively correlated with TE-SF, while price return is negatively correlated with TE-SF. Although higher price return reduces the TE-SF, it can increase the total return as evident from Equation (4); the first term in the right-hand side of this equation becomes larger for higher price return. Please note there could be a dividend growth and investment horizon where reducing price returns increases the wealth which the author will elaborate on in the near future. However, higher dividend growth, all else being equal, always leads to larger TE-SF.



(a) $n = 5\ years$



(b) $n = 10\ years$

Figure 7: TE-SF vs price return for different initial yield and investment horizon.

Higher price return reduces the TE-SF while greater yield increase always increases the TE-SF

Finally, it is worth emphasizing that the effectiveness of the exponential nature of the dividend growth comes through patience and over long investment horizon; see Figure 5. A long investment horizon and the fastest dividend growth are the mutually inclusive components engraved in the Bristol Gate investment thesis.

Conclusions:

It is mathematically proven that dividend growth's (income growth's) double exponential behavior improves total return. Numerical simulations are performed to study the relationship between this double exponential factor and initial yield, dividend growth, length of investment and price return. It is shown that the double exponential contributions increase by initial yield, investment horizon and the dividend growth and decrease by price return. Therefore, all else being equal, dividend growth has a positive double exponential impact on total return on a long investment horizon.

It is demonstrated that the reconstitution of a portfolio of high dividend growers, with a given frequency, e.g., five years, could lessen the concern of high dividend yield exposures while still maintaining the double exponential power of dividend growth. An example of such a limitation is the increase of the initial yield 2% to 16.27% when dividend growth is 15%, investment horizon is 15 year and price return is 0%.

The differential power of dividend growth is more sensible as the price return is close to zero or even slightly negative. This could be of particular interest for the investment regime where the consensus is conservative, and the market is bearish.

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